ON CERTAIN MORITA INVARIANTS INVOLVING COMMUTATOR SUBSPACE AND RADICAL POWERS

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ABSTRACT. For a finite-dimensional algebra A, we define $k(A)$ to be the codimension of the commutator subspace $K(A)$ and characterize algebras with small $k(A)$ up to Morita equivalence. This is achieved by extending Okuyama's refinement of Brandt's theorem to this setting. To this end, we study the codimension of the sum of the commutator subspace $K(A)$ and nth Jacobson radical $\mathrm{Rad}^n(A)$. We prove that this is Morita invariant and give an upper bound for the codimension as well. This is a report of a talk based on Koshitani and Sakurai [arXiv:1803.00025v2 (2018) 9pp.].

Key Words: Codimension, Commutator subspace, Finite-dimensional algebra, Morita invariant, Morita equivalence.

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1. INTRODUCTION

"For every positive integer n, there are only finitely many isomorphism classes of finite groups with n conjugacy classes."

This is a theorem due to E. Landau. Therefore structure of a finite group G can be studied by the number of conjugacy classes $k(G)$, as it can be studied by its order $|G|$. Let us see some small examples:

$$\begin{aligned} k(G) = 1 &\iff G \cong 1, \\ k(G) = 2 &\iff G \cong \mathbb{Z}/2\mathbb{Z}, \\ k(G) = 3 &\iff G \cong \mathbb{Z}/3\mathbb{Z}, \mathfrak{S}_3. \end{aligned} \tag{1.1}$$

According to a fundamental theorem in representation theory of finite groups, the set of conjugacy classes of a finite group G bijectively correspond to the set of irreducible characters $\mathrm{Irr}(G)$ of G. (Namely, $k(G) = |\mathrm{Irr}(G)|$.) Therefore, for every positive integer n, there are only finitely many isomorphism classes of finite groups with n irreducible characters. By the way, if a prime number p is provided, $\mathrm{Irr}(G)$ is partitioned into so-called p-blocks. Furthermore, some representation-theoretic properties are also naturally partitioned according to blocks and it is expected that something similar also holds for blocks.

This report is organized as follows. In the first section, character-theoretic background is presented. Analogous results for block algebras and some preceding studies are presented

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in the next section. In the last section, we present results generalized to arbitrary finite-dimensional algebras. These are obtained as corollaries of an extension of a result by Okuyama [8].

2. Character theory

Let us introduce a partition of irreducible characters into p-blocks.

Definition 1 ([7, pp. 62–63])**.** For a finite group G and a prime number p, let us define an undirected graph with the vertex set $\mathrm{Irr}(G)$ and two vertices $\chi, \psi \in \mathrm{Irr}(G)$ are adjacent if

$$\frac{1}{|G|} \sum_{\substack{g \in G \\ p \nmid |g|}} \chi(g)\overline{\psi(g)} \neq 0.$$

Then the partition of $\mathrm{Irr}(G)$ into its connected components is written as

$$\mathrm{Irr}(G) = \coprod_{B \in \mathrm{Bl}_p(G)} \mathrm{Irr}(B). \tag{2.1}$$

This is called the partition of $\mathrm{Irr}(G)$ into *p-blocks*. The block $B_0 \in \mathrm{Bl}_p(G)$ containing the trivial character 1_G is called the *principal block*. Define $k(B) = |\mathrm{Irr}(B)|$ for $B \in \mathrm{Bl}_p(G)$.

There is another way to measure how complex $\mathrm{Irr}(B)$ is other than $k(B)$, called a defect.

Definition 2 (defect)**.** Let ν_p be the p-adic valuation. Define $d(\chi) = \nu_p(|G|/\chi(1))$ for a character $\chi \in \mathrm{Irr}(G)$ and define $d(B) = \max_{\chi \in \mathrm{Irr}(B)} d(\chi)$ for a block $B \in \mathrm{Bl}_p(G)$. These are called the *p-defect* of χ and B, respectively.

For example, the order of Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$ can be given using the p-defect of the principal block $B_0 \in \mathrm{Bl}_p(G)$ by $|P| = p^{d(B_0)}$. The analogous results of (1.1) for blocks are the followings.

Theorem 3 (Brauer-Nesbitt [2], Blocks of Defect Zero)**.**

$$k(B) = 1 \iff p^{d(B)} = 1.$$

Theorem 4 (Brandt [1])**.**

$$k(B) = 2 \iff p^{d(B)} = 2.$$

It is expected that similar result holds for $k(B) = 3$, but unfortunately it is still open.

Conjecture 5 (cf. Brandt [1])**.**

$$k(B) = 3 \overset{?}{\iff} p^{d(B)} = 3.$$

More generally, it is conjectured that $k(B)$ does not exceed $p^{d(B)}$. This is a famous open problem which has been known since more than half a century ago.

Conjecture 6 ($k(B)$-Conjecture)**.**

$$k(B) \overset{?}{\leq} p^{d(B)}.$$

3. Block algebras

In this section, we assign an algebra to a block $B \in \mathrm{Bl}_p(G)$. We present how structure of such algebras are restricted if $k(B)$ are small.

Definition 7 (Osima idempotent)**.** Define $e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g$ for a character $\chi \in \mathrm{Irr}(G)$. This is a centrally primitive idempotent of the group algebra $\mathbb{C}G$. Define

$$f_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi$$

for a block $B \in \mathrm{Bl}_p(G)$. This is called the *Osima idempotent.*

Let F be the algebraic closure of the prime field $\mathbb{F}_p$ of characteristic $p > 0$. We can obtain a centrally primitive idempotent e_B of the group algebra FG by 'reduction modulo p' of the Osima idempotent f_B. (See [7, Chapters 2–3] for details.) This is called the *block idempotent* of B.

Definition 8 (block algebra)**.** For a block idempotent $e_B \in FG$, the algebra $e_B FG$ is called a *block algebra.*

Let $\mathsf{mod}\, A$ denote the category of finitely generated right A-modules for a finite-dimensional algebra A. The following propositions are basic ones.

Proposition 9.

$$\dim Z(e_B FG) = k(B).$$

Theorem 10 (cf. Theorems 3, 4)**.**

$$k(B) = 1 \iff \mathsf{mod}\, e_B FG \simeq \mathsf{mod}\, F,$$
$$k(B) = 2 \iff \mathsf{mod}\, e_B FG \simeq \mathsf{mod}\, F[X]/(X^2).$$

Okuyama [8] obtained the following theorem by refining a theorem of Brandt [1].

Definition 11. Define $\mathrm{Soc}^n(A) = \{\, x \in A \mid x \,\mathrm{Rad}^n(A) = 0 \,\}$ for a finite-dimensional algebra A. This is called *socle series* of A. Set $ZS^n(A) = Z(A) \cap \mathrm{Soc}^n(A)$.

Theorem 12 (Okuyama [8])**.** *Let $\{\, S_i \mid 1 \le i \le \ell(B) \,\}$ be a complete set of representatives of simple $e_B FG$-modules. Then*

$$\dim ZS^2(e_B FG) = \ell(B) + \sum_{i=1}^{\ell(B)} \dim \mathrm{Ext}^1_{e_B FG}(S_i, S_i).$$

Recently Otokita [9] extended this result as follows.

Theorem 13 (Otokita [9])**.** *Let $\{\, e_i \mid 1 \le i \le \ell(B) \,\}$ be a basic set of primitive idempotents of $e_B FG$. Then*

$$\dim ZS^n(e_B FG) \le \sum_{i=1}^{\ell(B)} \dim e_i(e_B FG)e_i / e_i \,\mathrm{Rad}^n(e_B FG) e_i$$

for every $n \ge 1$.

In the last section, we present generalizations of these theorems.

4. Results

In the following, most parts go well for fields other than the algebraically closed field F as long as it is large enough (i.e., splitting field) and even positive characteristic is not necessary. For brevity, however, we content ourselves with the field F.

Definition 14 (commutator subspace)**.** For a finite-dimensional algebra A over the field F, define

$$K(A) = \sum_{x,y\in A} F(xy - yx).$$

This is called the *commutator subspace* of A and define $k(A) = \operatorname{codim} K(A) = \dim A/K(A)$. Set $KR^n(A) = K(A) + \operatorname{Rad}^n(A)$.

Remark 15. Since $\dim Z(e_B FG) = \operatorname{codim} K(e_B FG)$ holds for a block algebra $e_B FG$, we have $k(B) = k(e_B FG)$. Hence our choice of notation is consistent.

Remark 16. The vector space $A/K(A)$ has many different names. It is called trace group $T(A)$, trace space $A/[A, A]$, zeroth Hochschild homology $HH_0(A)$ or zeroth cyclic homology $HC_0(A)$ of A in [4, 5, 11].

Theorem 17 (Koshitani-Sakurai [6])**.** *For every $n \geq 1$, $\operatorname{codim} KR^n(A)$ is a Morita invariant for a finite-dimensional algebra A.*

Taking Remark 15 into account, one can extend Theorems 13 and 12 as follows.

Theorem 18 (Koshitani-Sakurai [6])**.** *Let $\{ e_i \mid 1 \leq i \leq \ell(A) \}$ be a basic set of primitive idempotents of a finite-dimensional algebra A. Then*

$$\operatorname{codim} KR^n(A) \leq \sum_{i=1}^{\ell(A)} \dim e_i A e_i / e_i \operatorname{Rad}^n(A) e_i$$

for every $n \geq 1$.

Theorem 19 (Koshitani-Sakurai [6], Shimizu [10])**.** *Let $\{ S_i \mid 1 \leq i \leq \ell(A) \}$ be a complete set of representatives of simple A-modules. Then*

$$\operatorname{codim} KR^2(A) = \ell(A) + \sum_{i=1}^{\ell(A)} \dim \operatorname{Ext}^1_A(S_i, S_i).$$

Remark 20. It is known that $\operatorname{codim} KR^1(A) = \ell(A)$.

As corollaries of Theorems 18 and 19, we have the following.

Theorem 21 (Koshitani-Sakurai [6])**.** *Let C_A be the Cartan matrix of a finite-dimensional algebra A. Then*

$$\ell(A) + \sum_{i=1}^{\ell(A)} \dim \operatorname{Ext}^1_A(S_i, S_i) \leq k(A) \leq \operatorname{tr} C_A.$$

Theorem 22 (Koshitani-Sakurai [6], Chlebowitz [3])**.**

$$k(A) = 1 \iff \mathsf{mod}\, A \simeq \mathsf{mod}\, F,$$

$$k(A) = 2 \text{ and } \ell(A) = 1 \iff \mathsf{mod}\, A \simeq \mathsf{mod}\, F[X]/(X^2).$$

Remark 23. Theorem 19 is obtained independently from Shimizu [10]. His proof in [10] is, as far as we understand, done by reducing the proposition to a proposition for pointed dual coalgebras and using the Taft-Wilson theorem. It is different from how we proved in [6]. Theorem 22 and several further cases were already studied by Chlebowitz [3]. See [6] for details. (Truncated polynomial algebras $F[X]/(X^n)$ are also characterized in a similar fashion there.)

Remark 24. Let Q be a finite acyclic quiver and I an admissible ideal of the path algebra FQ. From Theorem 21, it can shown that $k(FQ/I) = \ell(FQ/I)$.

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